Section 9.1 Sequences

- **List the terms of a sequence.**
- **Determine whether a sequence converges or diverges.**
- **Write a formula for the *n*th term of a sequence.**
- **Use properties of monotonic sequences and bounded sequences.**

Finding Patterns Describe a pattern for each of the following sequences. Then use your description to write a formula for the nth term of each sequence. As n increases, do the terms appear to be approaching a limit? Explain your reasoning.

a. $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

b. $1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \ldots$

c. $10, \frac{10}{3}, \frac{10}{6}, \frac{10}{10}, \frac{10}{15}, \ldots$

d. $\frac{1}{4}, \frac{4}{9}, \frac{9}{16}, \frac{16}{25}, \frac{25}{36}, \ldots$

e. $\frac{3}{7}, \frac{5}{10}, \frac{7}{13}, \frac{9}{16}, \frac{11}{19}, \ldots$

Sequences

In mathematics, the word "sequence" is used in much the same way as in ordinary English. To say that a collection of objects or events is *in sequence* usually means that the collection is ordered so that it has an identified first member, second member, third member, and so on.

Mathematically, a **sequence** is defined as a function whose domain is the set of positive integers. Although a sequence is a function, it is common to represent sequences by subscript notation rather than by the standard function notation. For instance, in the sequence

$$\begin{array}{cccccccc} 1, & 2, & 3, & 4, & \ldots, & n, & \ldots & \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \text{Sequence} \\ a_1, & a_2, & a_3, & a_4, & \ldots, & a_n, & \ldots & \end{array}$$

1 is mapped onto a_1, 2 is mapped onto a_2, and so on. The numbers $a_1, a_2, a_3, \ldots, a_n, \ldots$ are the **terms** of the sequence. The number a_n is the ***n*th term** of the sequence, and the entire sequence is denoted by $\{a_n\}$.

NOTE Occasionally, it is convenient to begin a sequence with a_0, so that the terms of the sequence become $a_0, a_1, a_2, a_3, \ldots, a_n, \ldots$

EXAMPLE 1 Listing the Terms of a Sequence

a. The terms of the sequence $\{a_n\} = \{3 + (-1)^n\}$ are

$$\begin{array}{cccc} 3 + (-1)^1, & 3 + (-1)^2, & 3 + (-1)^3, & 3 + (-1)^4, \ldots \\ 2, & 4, & 2, & 4, \quad \ldots . \end{array}$$

b. The terms of the sequence $\{b_n\} = \left\{\dfrac{n}{1 - 2n}\right\}$ are

$$\begin{array}{cccc} \dfrac{1}{1 - 2 \cdot 1}, & \dfrac{2}{1 - 2 \cdot 2}, & \dfrac{3}{1 - 2 \cdot 3}, & \dfrac{4}{1 - 2 \cdot 4}, \ldots \\ -1, & -\dfrac{2}{3}, & -\dfrac{3}{5}, & -\dfrac{4}{7}, \quad \ldots . \end{array}$$

c. The terms of the sequence $\{c_n\} = \left\{\dfrac{n^2}{2^n - 1}\right\}$ are

$$\begin{array}{cccc} \dfrac{1^2}{2^1 - 1}, & \dfrac{2^2}{2^2 - 1}, & \dfrac{3^2}{2^3 - 1}, & \dfrac{4^2}{2^4 - 1}, \ldots \\ \dfrac{1}{1}, & \dfrac{4}{3}, & \dfrac{9}{7}, & \dfrac{16}{15}, \quad \ldots . \end{array}$$

STUDY TIP Some sequences are defined recursively. To define a sequence recursively, you need to be given one or more of the first few terms. All other terms of the sequence are then defined using previous terms, as shown in Example 1(d).

d. The terms of the **recursively defined** sequence $\{d_n\}$, where $d_1 = 25$ and $d_{n+1} = d_n - 5$ are

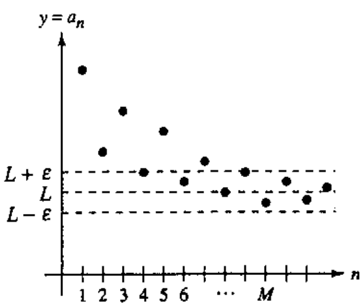
$$25, \quad 25 - 5 = 20, \quad 20 - 5 = 15, \quad 15 - 5 = 10, \ldots .$$

Limit of a Sequence

The primary focus of this chapter concerns sequences whose terms approach limiting values. Such sequences are said to **converge.** For instance, the sequence $\{1/2^n\}$

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$$

converges to 0, as indicated in the following definition.



For $n > M$, the terms of the sequence all lie within ε units of L.
Figure 9.1

Definition of the Limit of a Sequence

Let L be a real number. The **limit** of a sequence $\{a_n\}$ is L, written as

$$\lim_{n\to\infty} a_n = L$$

if for each $\varepsilon > 0$, there exists $M > 0$ such that $|a_n - L| < \varepsilon$ whenever $n > M$. If the limit L of a sequence exists, then the sequence **converges** to L. If the limit of a sequence does not exist, then the sequence **diverges.**

Graphically, this definition says that eventually (for $n > M$ and $\varepsilon > 0$) the terms of a sequence that converges to L will lie within the band between the lines $y = L + \varepsilon$ and $y = L - \varepsilon$, as shown in Figure 9.1.

If a sequence $\{a_n\}$ agrees with a function f at every positive integer, and if $f(x)$ approaches a limit L as $x \to \infty$, the sequence must converge to the same limit L.

THEOREM 9.1 Limit of a Sequence

Let L be a real number. Let f be a function of a real variable such that

$$\lim_{x\to\infty} f(x) = L.$$

If $\{a_n\}$ is a sequence such that $f(n) = a_n$ for every positive integer n, then

$$\lim_{n\to\infty} a_n = L.$$

EXAMPLE 2 Finding the Limit of a Sequence

Find the limit of the sequence whose nth term is

$$a_n = \left(1 + \frac{1}{n}\right)^n.$$

Solution In Theorem 5.15, you learned that

$$\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e.$$

So, you can apply Theorem 9.1 to conclude that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e.$$

NOTE There are different ways in which a sequence can fail to have a limit. One way is that the terms of the sequence increase without bound or decrease without bound. These cases are written symbolically as follows.

Terms increase without bound:

$$\lim_{n\to\infty} a_n = \infty$$

Terms decrease without bound:

$$\lim_{n\to\infty} a_n = -\infty$$

The following properties of limits of sequences parallel those given for limits of functions of a real variable in Section 1.3.

THEOREM 9.2 Properties of Limits of Sequences

Let $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = K$.

1. $\lim_{n\to\infty} (a_n \pm b_n) = L \pm K$
2. $\lim_{n\to\infty} ca_n = cL$, c is any real number
3. $\lim_{n\to\infty} (a_nb_n) = LK$
4. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{K}$, $b_n \neq 0$ and $K \neq 0$

EXAMPLE 3 Determining Convergence or Divergence

a. Because the sequence $\{a_n\} = \{3 + (-1)^n\}$ has terms

$$2, 4, 2, 4, . . .$$

See Example 1(a), page 594.

that alternate between 2 and 4, the limit

$$\lim_{n\to\infty} a_n$$

does not exist. So, the sequence diverges.

b. For $\{b_n\} = \left\{\frac{n}{1-2n}\right\}$, divide the numerator and denominator by n to obtain

$$\lim_{n\to\infty} \frac{n}{1-2n} = \lim_{n\to\infty} \frac{1}{(1/n) - 2} = -\frac{1}{2}$$

See Example 1(b), page 594.

which implies that the sequence converges to $-\frac{1}{2}$.

EXAMPLE 4 Using L'Hôpital's Rule to Determine Convergence

Show that the sequence whose nth term is $a_n = \frac{n^2}{2^n - 1}$ converges.

Solution Consider the function of a real variable

$$f(x) = \frac{x^2}{2^x - 1}.$$

Applying L'Hôpital's Rule twice produces

$$\lim_{x\to\infty} \frac{x^2}{2^x - 1} = \lim_{x\to\infty} \frac{2x}{(\ln 2)2^x} = \lim_{x\to\infty} \frac{2}{(\ln 2)^2 2^x} = 0.$$

Because $f(n) = a_n$ for every positive integer, you can apply Theorem 9.1 to conclude that

$$\lim_{n\to\infty} \frac{n^2}{2^n - 1} = 0.$$

See Example 1(c), page 594.

So, the sequence converges to 0.

TECHNOLOGY Use a graphing utility to graph the function in Example 4. Notice that as x approaches infinity, the value of the function gets closer and closer to 0. If you have access to a graphing utility that can generate terms of a sequence, try using it to calculate the first 20 terms of the sequence in Example 4. Then view the terms to observe numerically that the sequence converges to 0.

indicates that in the HM mathSpace® *CD-ROM and the online* Eduspace® *system for this text, you will find an Open Exploration, which further explores this example using the computer algebra systems* Maple, Mathcad, Mathematica, *and* Derive.

The symbol $n!$ (read "n factorial") is used to simplify some of the formulas developed in this chapter. Let n be a positive integer; then $\boldsymbol{n}$ **factorial** is defined as

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdot \cdot (n-1) \cdot n.$$

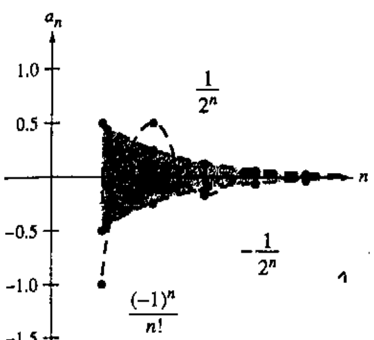
As a special case, **zero factorial** is defined as $0! = 1$. From this definition, you can see that $1! = 1$, $2! = 1 \cdot 2 = 2$, $3! = 1 \cdot 2 \cdot 3 = 6$, and so on. Factorials follow the same conventions for order of operations as exponents. That is, just as $2x^3$ and $(2x)^3$ imply different orders of operations, $2n!$ and $(2n)!$ imply the following orders.

$$2n! = 2(n!) = 2(1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdot \cdot n)$$

and

$$(2n)! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdot \cdot n \cdot (n+1) \cdot \cdot \cdot 2n$$

Another useful limit theorem that can be rewritten for sequences is the Squeeze Theorem from Section 1.3.



For $n \geq 4$, $(-1)^n/n!$ is squeezed between $-1/2^n$ and $1/2^n$.
Figure 9.2

THEOREM 9.3 Squeeze Theorem for Sequences

If

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} b_n$$

and there exists an integer N such that $a_n \leq c_n \leq b_n$ for all $n > N$, then

$$\lim_{n\to\infty} c_n = L.$$

EXAMPLE 5 Using the Squeeze Theorem

Show that the sequence $\{c_n\} = \left\{(-1)^n \frac{1}{n!}\right\}$ converges, and find its limit.

Solution To apply the Squeeze Theorem, you must find two convergent sequences that can be related to the given sequence. Two possibilities are $a_n = -1/2^n$ and $b_n = 1/2^n$, both of which converge to 0. By comparing the term $n!$ with 2^n, you can see that

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot \cdot \cdot n = 24 \cdot \underbrace{5 \cdot 6 \cdot \cdot \cdot n}_{n-4 \text{ factors}} \qquad (n \geq 4)$$

and

$$2^n = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot \cdot \cdot 2 = 16 \cdot \underbrace{2 \cdot 2 \cdot \cdot \cdot 2}_{n-4 \text{ factors}}. \qquad (n \geq 4)$$

This implies that for $n \geq 4$, $2^n < n!$, and you have

$$\frac{-1}{2^n} \leq (-1)^n \frac{1}{n!} \leq \frac{1}{2^n}, \quad n \geq 4$$

as shown in Figure 9.2. So, by the Squeeze Theorem it follows that

$$\lim_{n\to\infty} (-1)^n \frac{1}{n!} = 0.$$

NOTE Example 5 suggests something about the rate at which $n!$ increases as $n \to \infty$. As Figure 9.2 suggests, both $1/2^n$ and $1/n!$ approach 0 as $n \to \infty$. Yet $1/n!$ approaches 0 so much faster than $1/2^n$ does that

$$\lim_{n\to\infty} \frac{1/n!}{1/2^n} = \lim_{n\to\infty} \frac{2^n}{n!} = 0.$$

In fact, it can be shown that for any fixed number k,

$$\lim_{n\to\infty} \frac{k^n}{n!} = 0.$$

This means that *the factorial function grows faster than any exponential function.*

In Example 5, the sequence $\{c_n\}$ has both positive and negative terms. For this sequence, it happens that the sequence of absolute values, $\{|c_n|\}$, also converges to 0. You can show this by the Squeeze Theorem using the inequality

$$0 \le \frac{1}{n!} \le \frac{1}{2^n}, \quad n \ge 4.$$

In such cases, it is often convenient to consider the sequence of absolute values—and then apply Theorem 9.4, which states that if the absolute value sequence converges to 0, the original signed sequence also converges to 0.

THEOREM 9.4 Absolute Value Theorem

For the sequence $\{a_n\}$, if

$$\lim_{n\to\infty} |a_n| = 0 \qquad \text{then} \qquad \lim_{n\to\infty} a_n = 0.$$

Proof Consider the two sequences $\{|a_n|\}$ and $\{-|a_n|\}$. Because both of these sequences converge to 0 and

$$-|a_n| \le a_n \le |a_n|$$

you can use the Squeeze Theorem to conclude that $\{a_n\}$ converges to 0.

Pattern Recognition for Sequences

Sometimes the terms of a sequence are generated by some rule that does not explicitly identify the nth term of the sequence. In such cases, you may be required to discover a *pattern* in the sequence and to describe the nth term. Once the nth term has been specified, you can investigate the convergence or divergence of the sequence.

EXAMPLE 6 Finding the *n*th Term of a Sequence

Find a sequence $\{a_n\}$ whose first five terms are

$$\frac{2}{1}, \frac{4}{3}, \frac{8}{5}, \frac{16}{7}, \frac{32}{9}, \ldots$$

and then determine whether the particular sequence you have chosen converges or diverges.

Solution First, note that the numerators are successive powers of 2, and the denominators form the sequence of positive odd integers. By comparing a_n with n, you have the following pattern.

$$\frac{2^1}{1}, \frac{2^2}{3}, \frac{2^3}{5}, \frac{2^4}{7}, \frac{2^5}{9}, \ldots, \frac{2^n}{2n-1}$$

Using L'Hôpital's Rule to evaluate the limit of $f(x) = 2^x/(2x-1)$, you obtain

$$\lim_{x\to\infty} \frac{2^x}{2x-1} = \lim_{x\to\infty} \frac{2^x(\ln 2)}{2} = \infty \quad \Longrightarrow \quad \lim_{n\to\infty} \frac{2^n}{2n-1} = \infty.$$

So, the sequence diverges.

Without a specific rule for generating the terms of a sequence or some knowledge of the context in which the terms of the sequence are obtained, it is not possible to determine the convergence or divergence of the sequence merely from its first several terms. For instance, although the first three terms of the following four sequences are identical, the first two sequences converge to 0, the third sequence converges to $\frac{1}{9}$, and the fourth sequence diverges.

$$\{a_n\}: \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots, \frac{1}{2^n}, \ldots$$

$$\{b_n\}: \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{15}, \ldots, \frac{6}{(n+1)(n^2-n+6)}, \ldots$$

$$\{c_n\}: \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{7}{62}, \ldots, \frac{n^2-3n+3}{9n^2-25n+18}, \ldots$$

$$\{d_n\}: \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, 0, \ldots, \frac{-n(n+1)(n-4)}{6(n^2+3n-2)}, \ldots$$

The process of determining an nth term from the pattern observed in the first several terms of a sequence is an example of *inductive reasoning*.

EXAMPLE 7 Finding the nth Term of a Sequence

Determine an nth term for a sequence whose first five terms are

$$-\frac{2}{1}, \frac{8}{2}, -\frac{26}{6}, \frac{80}{24}, -\frac{242}{120}, \ldots$$

and then decide whether the sequence converges or diverges.

Solution Note that the numerators are 1 less than 3^n. So, you can reason that the numerators are given by the rule $3^n - 1$. Factoring the denominators produces

$$\begin{aligned} 1 &= 1 \\ 2 &= 1 \cdot 2 \\ 6 &= 1 \cdot 2 \cdot 3 \\ 24 &= 1 \cdot 2 \cdot 3 \cdot 4 \\ 120 &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot \cdots. \end{aligned}$$

This suggests that the denominators are represented by $n!$. Finally, because the signs alternate, you can write the nth term as

$$a_n = (-1)^n\left(\frac{3^n - 1}{n!}\right).$$

From the discussion about the growth of $n!$, it follows that

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{3^n - 1}{n!} = 0.$$

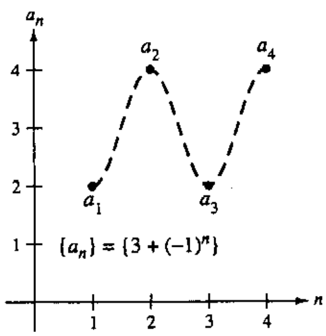
Applying Theorem 9.4, you can conclude that

$$\lim_{n\to\infty} a_n = 0.$$

So, the sequence $\{a_n\}$ converges to 0.

Monotonic Sequences and Bounded Sequences

So far you have determined the convergence of a sequence by finding its limit. Even if you cannot determine the limit of a particular sequence, it still may be useful to know whether the sequence converges. Theorem 9.5 provides a test for convergence of sequences without determining the limit. First, some preliminary definitions are given.



(a) Not monotonic

Definition of a Monotonic Sequence

A sequence $\{a_n\}$ is **monotonic** if its terms are nondecreasing

$$a_1 \le a_2 \le a_3 \le \cdots \le a_n \le \cdots$$

or if its terms are nonincreasing

$$a_1 \ge a_2 \ge a_3 \ge \cdots \ge a_n \ge \cdots.$$

EXAMPLE 8 Determining Whether a Sequence Is Monotonic

Determine whether each sequence having the given nth term is monotonic.

a. $a_n = 3 + (-1)^n$ **b.** $b_n = \dfrac{2n}{1+n}$ **c.** $c_n = \dfrac{n^2}{2^n - 1}$

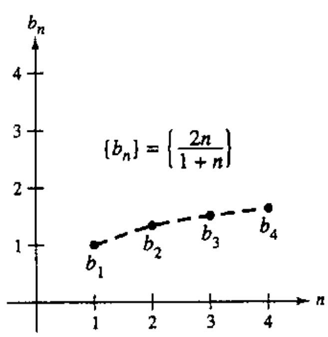
Solution

a. This sequence alternates between 2 and 4. So, it is not monotonic.

b. This sequence is monotonic because each successive term is larger than its predecessor. To see this, compare the terms b_n and b_{n+1}. [Note that, because n is positive, you can multiply each side of the inequality by $(1 + n)$ and $(2 + n)$ without reversing the inequality sign.]

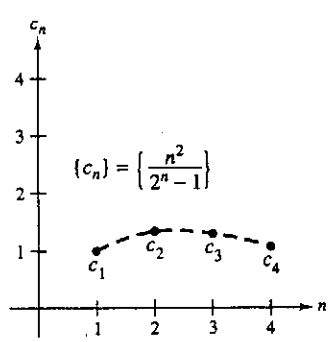
$$b_n = \frac{2n}{1+n} \overset{?}{<} \frac{2(n+1)}{1+(n+1)} = b_{n+1}$$
$$2n(2+n) \overset{?}{<} (1+n)(2n+2)$$
$$4n + 2n^2 \overset{?}{<} 2 + 4n + 2n^2$$
$$0 < 2$$

Starting with the final inequality, which is valid, you can reverse the steps to conclude that the original inequality is also valid.



(b) Monotonic

c. This sequence is not monotonic, because the second term is larger than the first term, and larger than the third. (Note that if you drop the first term, the remaining sequence $c_2, c_3, c_4, \ldots$ is monotonic.)



(c) Not monotonic
Figure 9.3

Figure 9.3 graphically illustrates these three sequences.

NOTE In Example 8(b), another way to see that the sequence is monotonic is to argue that the derivative of the corresponding differentiable function $f(x) = 2x/(1 + x)$ is positive for all x. This implies that f is increasing, which in turn implies that $\{a_n\}$ is increasing.

NOTE All three sequences shown in Figure 9.3 are bounded. To see this, consider the following.

$$2 \le a_n \le 4$$
$$1 \le b_n \le 2$$
$$0 \le c_n \le \frac{4}{3}$$

Definition of a Bounded Sequence

1. A sequence $\{a_n\}$ is **bounded above** if there is a real number M such that $a_n \le M$ for all n. The number M is called an **upper bound** of the sequence.
2. A sequence $\{a_n\}$ is **bounded below** if there is a real number N such that $N \le a_n$ for all n. The number N is called a **lower bound** of the sequence.
3. A sequence $\{a_n\}$ is **bounded** if it is bounded above and bounded below.

One important property of the real numbers is that they are **complete.** Informally, this means that there are no holes or gaps on the real number line. (The set of rational numbers does not have the completeness property.) The completeness axiom for real numbers can be used to conclude that if a sequence has an upper bound, it must have a **least upper bound** (an upper bound that is smaller than all other upper bounds for the sequence). For example, the least upper bound of the sequence $\{a_n\} = \{n/(n+1)\}$,

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots$$

is 1. The completeness axiom is used in the proof of Theorem 9.5.

THEOREM 9.5 Bounded Monotonic Sequences

If a sequence $\{a_n\}$ is bounded and monotonic, then it converges.

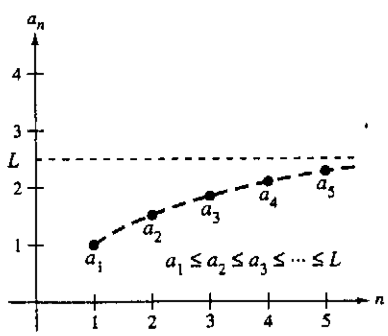
Proof Assume that the sequence is nondecreasing, as shown in Figure 9.4. For the sake of simplicity, also assume that each term in the sequence is positive. Because the sequence is bounded, there must exist an upper bound M such that

$$a_1 \le a_2 \le a_3 \le \cdots \le a_n \le \cdots \le M.$$

From the completeness axiom, it follows that there is a least upper bound L such that

$$a_1 \le a_2 \le a_3 \le \cdots \le a_n \le \cdots \le L.$$

For $\varepsilon > 0$, it follows that $L - \varepsilon < L$, and therefore $L - \varepsilon$ cannot be an upper bound for the sequence. Consequently, at least one term of $\{a_n\}$ is greater than $L - \varepsilon$. That is, $L - \varepsilon < a_N$ for some positive integer N. Because the terms of $\{a_n\}$ are nondecreasing, it follows that $a_N \le a_n$ for $n > N$. You now know that $L - \varepsilon < a_N \le a_n \le L < L + \varepsilon$, for every $n > N$. It follows that $|a_n - L| < \varepsilon$ for $n > N$, which by definition means that $\{a_n\}$ converges to L. The proof for a nonincreasing sequence is similar.



Every bounded nondecreasing sequence converges.
Figure 9.4

EXAMPLE 9 Bounded and Monotonic Sequences

a. The sequence $\{a_n\} = \{1/n\}$ is both bounded and monotonic and so, by Theorem 9.5, must converge.

b. The divergent sequence $\{b_n\} = \{n^2/(n+1)\}$ is monotonic, but not bounded. (It is bounded below.)

c. The divergent sequence $\{c_n\} = \{(-1)^n\}$ is bounded, but not monotonic.

Exercises for Section 9.1

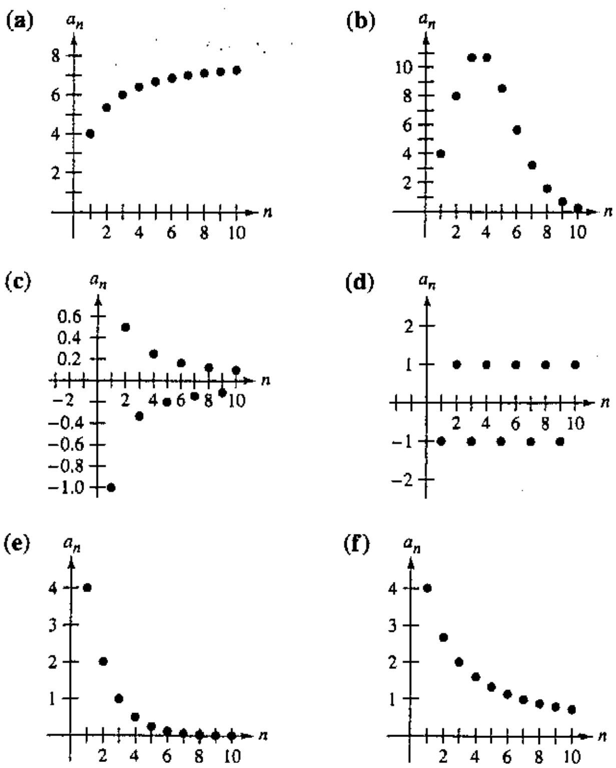
In Exercises 1–10, write the first five terms of the sequence.

1. $a_n = 2^n$
2. $a_n = \dfrac{3^n}{n!}$
3. $a_n = \left(-\frac{1}{2}\right)^n$
4. $a_n = \left(-\frac{2}{3}\right)^n$
5. $a_n = \sin \dfrac{n\pi}{2}$
6. $a_n = \dfrac{2n}{n+3}$
7. $a_n = \dfrac{(-1)^{n(n+1)/2}}{n^2}$
8. $a_n = (-1)^{n+1}\left(\dfrac{2}{n}\right)$
9. $a_n = 5 - \dfrac{1}{n} + \dfrac{1}{n^2}$
10. $a_n = 10 + \dfrac{2}{n} + \dfrac{6}{n^2}$

In Exercises 11–14, write the first five terms of the recursively defined sequence.

11. $a_1 = 3,\ a_{k+1} = 2(a_k - 1)$
12. $a_1 = 4,\ a_{k+1} = \left(\dfrac{k+1}{2}\right)a_k$
13. $a_1 = 32,\ a_{k+1} = \frac{1}{2}a_k$
14. $a_1 = 6,\ a_{k+1} = \frac{1}{3}a_k^2$

In Exercises 15–20, match the sequence with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]



15. $a_n = \dfrac{8}{n+1}$
16. $a_n = \dfrac{8n}{n+1}$
17. $a_n = 4(0.5)^{n-1}$
18. $a_n = \dfrac{4^n}{n!}$
19. $a_n = (-1)^n$
20. $a_n = \dfrac{(-1)^n}{n}$

In Exercises 21–24, use a graphing utility to graph the first 10 terms of the sequence.

21. $a_n = \dfrac{2}{3}n$
22. $a_n = 2 - \dfrac{4}{n}$
23. $a_n = 16(-0.5)^{n-1}$
24. $a_n = \dfrac{2n}{n+1}$

In Exercises 25–30, write the next two apparent terms of the sequence. Describe the pattern you used to find these terms.

25. $2, 5, 8, 11, \ldots$
26. $\frac{7}{2}, 4, \frac{9}{2}, 5, \ldots$
27. $5, 10, 20, 40, \ldots$
28. $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \ldots$
29. $3, -\frac{3}{2}, \frac{3}{4}, -\frac{3}{8}, \ldots$
30. $1, -\frac{3}{2}, \frac{9}{4}, -\frac{27}{8}, \ldots$

In Exercises 31–36, simplify the ratio of factorials.

31. $\dfrac{10!}{8!}$
32. $\dfrac{25!}{23!}$
33. $\dfrac{(n+1)!}{n!}$
34. $\dfrac{(n+2)!}{n!}$
35. $\dfrac{(2n-1)!}{(2n+1)!}$
36. $\dfrac{(2n+2)!}{(2n)!}$

In Exercises 37–42, find the limit (if possible) of the sequence.

37. $a_n = \dfrac{5n^2}{n^2+2}$
38. $a_n = 5 - \dfrac{1}{n^2}$
39. $a_n = \dfrac{2n}{\sqrt{n^2+1}}$
40. $a_n = \dfrac{5n}{\sqrt{n^2+4}}$
41. $a_n = \sin \dfrac{1}{n}$
42. $a_n = \cos \dfrac{2}{n}$

In Exercises 43–46, use a graphing utility to graph the first 10 terms of the sequence. Use the graph to make an inference about the convergence or divergence of the sequence. Verify your inference analytically and, if the sequence converges, find its limit.

43. $a_n = \dfrac{n+1}{n}$
44. $a_n = \dfrac{1}{n^{3/2}}$
45. $a_n = \cos \dfrac{n\pi}{2}$
46. $a_n = 3 - \dfrac{1}{2^n}$

In Exercises 47–68, determine the convergence or divergence of the sequence with the given nth term. If the sequence converges, find its limit.

47. $a_n = (-1)^n\left(\dfrac{n}{n+1}\right)$
48. $a_n = 1 + (-1)^n$
49. $a_n = \dfrac{3n^2 - n + 4}{2n^2 + 1}$
50. $a_n = \dfrac{\sqrt[3]{n}}{\sqrt[3]{n} + 1}$
51. $a_n = \dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{(2n)^n}$
52. $a_n = \dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{n!}$

53. $a_n = \dfrac{1 + (-1)^n}{n}$

54. $a_n = \dfrac{1 + (-1)^n}{n^2}$

55. $a_n = \dfrac{\ln(n^3)}{2n}$

56. $a_n = \dfrac{\ln \sqrt{n}}{n}$

57. $a_n = \dfrac{3^n}{4^n}$

58. $a_n = (0.5)^n$

59. $a_n = \dfrac{(n + 1)!}{n!}$

60. $a_n = \dfrac{(n - 2)!}{n!}$

61. $a_n = \dfrac{n - 1}{n} - \dfrac{n}{n - 1}, \quad n \geq 2$

62. $a_n = \dfrac{n^2}{2n + 1} - \dfrac{n^2}{2n - 1}$

63. $a_n = \dfrac{n^p}{e^n}, \quad p > 0$

64. $a_n = n \sin \dfrac{1}{n}$

65. $a_n = \left(1 + \dfrac{k}{n}\right)^n$

66. $a_n = 2^{1/n}$

67. $a_n = \dfrac{\sin n}{n}$

68. $a_n = \dfrac{\cos \pi n}{n^2}$

In Exercises 69–82, write an expression for the *n*th term of the sequence. (There is more than one correct answer.)

69. $1, 4, 7, 10, \ldots$

70. $3, 7, 11, 15, \ldots$

71. $-1, 2, 7, 14, 23, \ldots$

72. $1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \ldots$

73. $\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots$

74. $2, -1, \frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, \ldots$

75. $2, 1 + \frac{1}{2}, 1 + \frac{1}{3}, 1 + \frac{1}{4}, 1 + \frac{1}{5}, \ldots$

76. $1 + \frac{1}{2}, 1 + \frac{3}{4}, 1 + \frac{7}{8}, 1 + \frac{15}{16}, 1 + \frac{31}{32}, \ldots$

77. $\dfrac{1}{2 \cdot 3}, \dfrac{2}{3 \cdot 4}, \dfrac{3}{4 \cdot 5}, \dfrac{4}{5 \cdot 6}, \ldots$

78. $1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \ldots$

79. $1, -\dfrac{1}{1 \cdot 3}, \dfrac{1}{1 \cdot 3 \cdot 5}, -\dfrac{1}{1 \cdot 3 \cdot 5 \cdot 7}, \ldots$

80. $1, x, \dfrac{x^2}{2}, \dfrac{x^3}{6}, \dfrac{x^4}{24}, \dfrac{x^5}{120}, \ldots$

81. $2, 24, 720, 40{,}320, 3{,}628{,}800, \ldots$

82. $1, 6, 120, 5040, 362{,}880, \ldots$

In Exercises 83–94, determine whether the sequence with the given *n*th term is monotonic. Discuss the boundedness of the sequence. Use a graphing utility to confirm your results.

83. $a_n = 4 - \dfrac{1}{n}$

84. $a_n = \dfrac{3n}{n + 2}$

85. $a_n = \dfrac{n}{2^{n+2}}$

86. $a_n = ne^{-n/2}$

87. $a_n = (-1)^n\left(\dfrac{1}{n}\right)$

88. $a_n = \left(-\dfrac{2}{3}\right)^n$

89. $a_n = \left(\dfrac{2}{3}\right)^n$

90. $a_n = \left(\dfrac{3}{2}\right)^n$

91. $a_n = \sin \dfrac{n\pi}{6}$

92. $a_n = \cos\left(\dfrac{n\pi}{2}\right)$

93. $a_n = \dfrac{\cos n}{n}$

94. $a_n = \dfrac{\sin \sqrt{n}}{n}$

In Exercises 95–98, (a) use Theorem 9.5 to show that the sequence with the given *n*th term converges and (b) use a graphing utility to graph the first 10 terms of the sequence and find its limit.

95. $a_n = 5 + \dfrac{1}{n}$

96. $a_n = 4 - \dfrac{3}{n}$

97. $a_n = \dfrac{1}{3}\left(1 - \dfrac{1}{3^n}\right)$

98. $a_n = 4 + \dfrac{1}{2^n}$

99. Let $\{a_n\}$ be an increasing sequence such that $2 \leq a_n \leq 4$. Explain why $\{a_n\}$ has a limit. What can you conclude about the limit?

100. Let $\{a_n\}$ be a monotonic sequence such that $a_n \leq 1$. Discuss the convergence of $\{a_n\}$. If $\{a_n\}$ converges, what can you conclude about its limit?

101. *Compound Interest* Consider the sequence $\{A_n\}$ whose *n*th term is given by

$$A_n = P\left(1 + \frac{r}{12}\right)^n$$

where P is the principal, A_n is the account balance after n months, and r is the interest rate compounded annually.

(a) Is $\{A_n\}$ a convergent sequence? Explain.

(b) Find the first 10 terms of the sequence if $P = \$9000$ and $r = 0.055$.

102. *Compound Interest* A deposit of \$100 is made at the beginning of each month in an account at an annual interest rate of 3% compounded monthly. The balance in the account after n months is $A_n = 100(401)(1.0025^n - 1)$.

(a) Compute the first six terms of the sequence $\{A_n\}$.

(b) Find the balance in the account after 5 years by computing the 60th term of the sequence.

(c) Find the balance in the account after 20 years by computing the 240th term of the sequence.

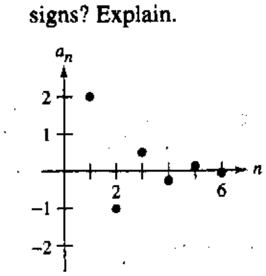
Writing About Concepts

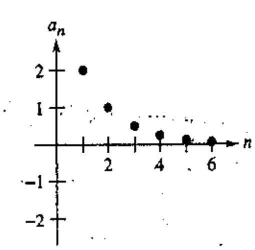
103. In your own words, define each of the following.

(a) Sequence (b) Convergence of a sequence

(c) Monotonic sequence (d) Bounded sequence

104. The graphs of two sequences are shown in the figures. Which graph represents the sequence with alternating signs? Explain.





Writing About Concepts *(continued)*

In Exercises 105–108, give an example of a sequence satisfying the condition or explain why no such sequence exists. (Examples are not unique.)

105. A monotonically increasing sequence that converges to 10

106. A monotonically increasing bounded sequence that does not converge

107. A sequence that converges to $\frac{3}{4}$

108. An unbounded sequence that converges to 100

109. ***Government Expenditures*** A government program that currently costs taxpayers \$2.5 billion per year is cut back by 20 percent per year.

(a) Write an expression for the amount budgeted for this program after n years.

(b) Compute the budgets for the first 4 years.

(c) Determine the convergence or divergence of the sequence of reduced budgets. If the sequence converges, find its limit.

110. ***Inflation*** If the rate of inflation is $4\frac{1}{2}\%$ per year and the average price of a car is currently \$16,000, the average price after n years is

$$P_n = \$16{,}000(1.045)^n.$$

Compute the average prices for the next 5 years.

111. ***Modeling Data*** The number a_n of endangered and threatened species in the United States from 1996 through 2002 is shown in the table, where n represents the year, with $n = 6$ corresponding to 1996. *(Source: U.S. Fish and Wildlife Service)*

n	6	7	8	9	10	11	12
a_n	1053	1132	1194	1205	1244	1254	1263

(a) Use the regression capabilities of a graphing utility to find a model of the form

$$a_n = bn^2 + cn + d, \qquad n = 6, 7, \ldots, 12$$

for the data. Use the graphing utility to plot the points and graph the model.

(b) Use the model to predict the number of endangered and threatened species in the year 2008.

112. ***Modeling Data*** The annual sales a_n (in millions of dollars) for Avon Products, Inc. from 1993 through 2002 are given below as ordered pairs of the form (n, a_n), where n represents the year, with $n = 3$ corresponding to 1993. *(Source: 2002 Avon Products, Inc. Annual Report)*

$(3, 3844), (4, 4267), (5, 4492), (6, 4814), (7, 5079),$
$(8, 5213), (9, 5289), (10, 5682), (11, 5958), (12, 6171)$

(a) Use the regression capabilities of a graphing utility to find a model of the form

$$a_n = bn + c, \quad n = 3, 4, \ldots, 12$$

for the data. Graphically compare the points and the model.

(b) Use the model to predict sales in the year 2008.

113. ***Comparing Exponential and Factorial Growth*** Consider the sequence $a_n = 10^n/n!$.

(a) Find two consecutive terms that are equal in magnitude.

(b) Are the terms following those found in part (a) increasing or decreasing?

(c) In Section 8.7, Exercises 65–70, it was shown that for "large" values of the independent variable an exponential function increases more rapidly than a polynomial function. From the result in part (b), what inference can you make about the rate of growth of an exponential function versus a factorial function for "large" integer values of n?

114. Compute the first six terms of the sequence

$$\{a_n\} = \left\{\left(1 + \frac{1}{n}\right)^n\right\}.$$

If the sequence converges, find its limit.

115. Compute the first six terms of the sequence $\{a_n\} = \{\sqrt[n]{n}\}$. If the sequence converges, find its limit.

116. Prove that if $\{s_n\}$ converges to L and $L > 0$, then there exists a number N such that $s_n > 0$ for $n > N$.

True or False? **In Exercises 117–120, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

117. If $\{a_n\}$ converges to 3 and $\{b_n\}$ converges to 2, then $\{a_n + b_n\}$ converges to 5.

118. If $\{a_n\}$ converges, then $\lim_{n\to\infty} (a_n - a_{n+1}) = 0$.

119. If $n > 1$, then $n! = n(n - 1)!$.

120. If $\{a_n\}$ converges, then $\{a_n/n\}$ converges to 0.

121. ***Fibonacci Sequence*** In a study of the progeny of rabbits, Fibonacci (ca. 1170–ca. 1240) encountered the sequence now bearing his name. It is defined recursively by

$$a_{n+2} = a_n + a_{n+1}, \quad \text{where} \quad a_1 = 1 \text{ and } a_2 = 1.$$

(a) Write the first 12 terms of the sequence.

(b) Write the first 10 terms of the sequence defined by

$$b_n = \frac{a_{n+1}}{a_n}, \quad n \ge 1.$$

(c) Using the definition in part (b), show that

$$b_n = 1 + \frac{1}{b_{n-1}}.$$

(d) The **golden ratio** ρ can be defined by $\lim_{n\to\infty} b_n = \rho$. Show that $\rho = 1 + 1/\rho$ and solve this equation for ρ.

122. ***Conjecture*** Let $x_0 = 1$ and consider the sequence x_n given by the formula

$$x_n = \frac{1}{2}x_{n-1} + \frac{1}{x_{n-1}}, \qquad n = 1, 2, \ldots .$$

Use a graphing utility to compute the first 10 terms of the sequence and make a conjecture about the limit of the sequence.

123. Consider the sequence

$$\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$$

(a) Compute the first five terms of this sequence.

(b) Write a recursion formula for a_n, for $n \geq 2$.

(c) Find $\lim_{n\to\infty} a_n$.

124. Consider the sequence

$$\sqrt{6}, \sqrt{6+\sqrt{6}}, \sqrt{6+\sqrt{6+\sqrt{6}}}, \ldots$$

(a) Compute the first five terms of this sequence.

(b) Write a recursion formula for a_n, for $n \geq 2$.

(c) Find $\lim_{n\to\infty} a_n$.

125. Consider the sequence $\{a_n\}$ where $a_1 = \sqrt{k}$, $a_{n+1} = \sqrt{k + a_n}$, and $k > 0$.

(a) Show that $\{a_n\}$ is increasing and bounded.

(b) Prove that $\lim_{n\to\infty} a_n$ exists.

(c) Find $\lim_{n\to\infty} a_n$.

126. ***Arithmetic-Geometric Mean*** Let $a_0 > b_0 > 0$. Let a_1 be the arithmetic mean of a_0 and b_0 and let b_1 be the geometric mean of a_0 and b_0.

$$a_1 = \frac{a_0 + b_0}{2} \qquad \text{Arithmetic mean}$$

$$b_1 = \sqrt{a_0 b_0} \qquad \text{Geometric mean}$$

Now define the sequences $\{a_n\}$ and $\{b_n\}$ as follows.

$$a_n = \frac{a_{n-1} + b_{n-1}}{2} \qquad b_n = \sqrt{a_{n-1}b_{n-1}}$$

(a) Let $a_0 = 10$ and $b_0 = 3$. Write out the first five terms of $\{a_n\}$ and $\{b_n\}$. Compare the terms of $\{b_n\}$. Compare a_n and b_n. What do you notice?

(b) Use induction to show that $a_n > a_{n+1} > b_{n+1} > b_n$, for $a_0 > b_0 > 0$.

(c) Explain why $\{a_n\}$ and $\{b_n\}$ are both convergent.

(d) Show that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

127. (a) Let $f(x) = \sin x$ and $a_n = n \sin 1/n$. Show that

$$\lim_{n\to\infty} a_n - f'(0) = 1.$$

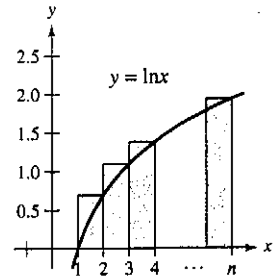
(b) Let $f(x)$ be differentiable on the interval $[0, 1]$ and $f(0) = 0$. Consider the sequence $\{a_n\}$, where $a_n = nf(1/n)$. Show that $\lim_{n\to\infty} a_n = f'(0)$.

128. Consider the sequence $\{a_n\} = \{nr^n\}$. Decide whether $\{a_n\}$ converges for each value of r.

(a) $r = \frac{1}{2}$ (b) $r = 1$ (c) $r = \frac{3}{2}$

(d) For what values or r does the sequence $\{nr^n\}$ converge?

129. (a) Show that $\int_1^n \ln x\,dx < \ln(n!)$ for $n \geq 2$.



(b) Draw a graph similar to the one above that shows

$$\ln(n!) < \int_1^{n+1} \ln x\,dx.$$

(c) Use the results of parts (a) and (b) to show that

$$\frac{n^n}{e^{n-1}} < n! < \frac{(n+1)^{n+1}}{e^n}, \text{ for } n > 1.$$

(d) Use the Squeeze Theorem for Sequences and the result of part (c) to show that

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}.$$

(e) Test the result of part (d) for $n = 20, 50$, and 100.

130. Consider the sequence $\{a_n\} = \left\{\frac{1}{n}\sum_{k=1}^{n} \frac{1}{1 + (k/n)}\right\}$.

(a) Write the first five terms of $\{a_n\}$.

(b) Show that $\lim_{n\to\infty} a_n = \ln 2$ by interpreting a_n as a Riemann sum of a definite integral.

131. Prove, using the definition of the limit of a sequence, that

$$\lim_{n\to\infty} \frac{1}{n^3} = 0.$$

132. Prove, using the definition of the limit of a sequence, that

$$\lim_{n\to\infty} r^n = 0 \text{ for } -1 < r < 1.$$

133. Complete the proof of Theorem 9.5.

Putnam Exam Challenge

134. Let $\{x_n\}$, $n \geq 0$, be a sequence of nonzero real numbers such that $x_n^2 - x_{n-1}x_{n+1} = 1$ for $n = 1, 2, 3, \ldots$. Prove that there exists a real number a such that $x_{n+1} = ax_n - x_{n-1}$, for all $n \geq 1$.

135. Let $T_0 = 2$, $T_1 = 3$, $T_2 = 6$, and, for $n \geq 3$,

$$T_n = (n+4)T_{n-1} - 4nT_{n-2} + (4n-8)T_{n-3}.$$

The first 10 terms of the sequence are

2, 3, 6, 14, 40, 152, 784, 5168, 40,576, 363,392.

Find, with proof, a formula for T_n of the form $T_n = A_n + B_n$, where $\{A_n\}$ and $\{B_n\}$ are well-known sequences.